Approximate Theoretical Performance Evaluation for a Diverging Rocket

By

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(With 4 Figures)

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Abstract — Zusammenfassung — Résumé

Approximate Theoretical Performance Evaluation for a Diverging Rocket. A simplified combustion model, which is motivated by available performance studies on the diverging rocket reactor, has been used as basis for an engine performance evaluation. Comparison with conventional rocket configurations shows that an upper performance limit for the diverging reactor is comparable with performance estimates for engines using an adiabatic work cycle. Development of the diverging reactor for engine applications may, however, offer some advantages for very hot, high-energy, propellant systems.

Näherungsweise theoretische Ermittlung der Wirkungsweise eines Raketenmotors mit durchwegs divergierender Düse. Es wurde ein vereinfachtes Modell der Verbrennung, welches aus verfügbaren Untersuchungen über das Verhalten eines Raketenmotors mit durchwegs divergierender Düse erhalten wurde, als Basis für die theoretische Ermittlung des Betriebsverhaltens eines solchen Triebwerkes verwendet. Ein Vergleich mit üblichen Raketenmotoren ergibt, daß ähnlich wie bei den üblichen adiabatischen Triebwerken eine obere Grenze für den Betrieb besteht. Die Entwicklung dieses Triebwerkes dürfte vor allem auf dem Gebiete hoher Temperaturen bei Verwendung hochenergetischer Treibstoffe liegen.

Evaluation théorique approchée des performances d'un moteur-fusée à chambre divergente. Un modèle simplifié pour la combustion, justifié par les études de performance connues, a été utilisé pour l'évaluation des chambres divergentes. La comparaison avec les configurations conventionnelles place la limite supérieure de performances au niveau des moteurs utilisant un cycle adiabatique. Le développement de la chambre divergente pour les moteurs-fusée peut néanmoins être avantageux avec les ergols à enthalpie spécifique et température de combustion très élevées.

I. Introduction

In previous publications we have discussed application of the diverging reactor for the determination of overall kinetic parameters under the conditions actually existing in rocket combustion chambers [1], [2]. In the present discussion we

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shall outline an approximate engine performance evaluation for a diverging rocket chamber[1]. As basis for this evaluation we choose a simplified combustion model that we have used previously for the computation of various dimensionless groups [1].

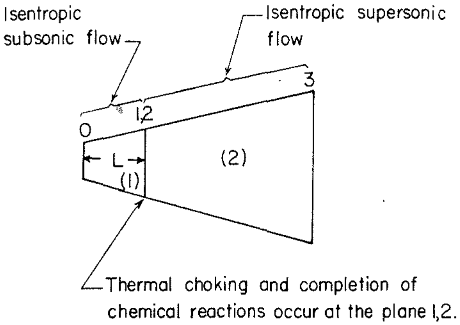

Fig. 1. Schematic diagram of a diverging rocket engine

A schematic diagram of the engine and combustion front is shown in Fig. 1. For the sake of simplicity, we assume that all of the heat release occurs at the plane 1, 2 in Fig. 1, that the gases behave as ideal gases with a constant ratio γ for the specific heat at constant pressure to the specific heat at constant volume, and that isentropic, one-dimensional, expansion occurs in the regions upstream and downstream from the reaction front. We are led to the model implicit in these approximations through the experimentally observed fact that the sonic plane occurs very close to the injector [1] and that the effective reaction rates are very fast (with low overall activation energy) near the injector end whereas they are relatively slow farther downstream (with large effective activation energy). Since the limited available experimental data suggest that most of the reactions are completed very close to the injector end, we may consider the present analysis with the characteristic length L in Fig. 1 equal to zero, or at least very small, to provide a reasonable upper limit for the performance which can be achieved with a diverging reactor.

II. Outline of Theoretical Considerations

1. *Pressure Ratio Across the Reaction Front*

The performance of a diverging rocket engine is a sensitive function of the stagnation pressure ratio p_{s2}/p_{s1} across the reaction plane. Fortunately it turns out that reasonable values of the heat release lead to values of p_{s2}/p_{s1} which are practically constant at about 0.8. We shall now prove the validity of this last statement by utilizing appropriate, simplified, versions of the conservation equations.

The overall conservation of mass equation at the plane 1, 2 in Fig. 1 is

$$\varrho_1 v_1 = \varrho_2 v_2$$

where ϱ and v identify, respectively, the density and linear flow velocity. Multiplying and dividing by $\sqrt{\gamma R_g T}$ and replacing ϱ by $p/R_g T$ we find that

$$\varrho v = \frac{p}{R_g T} \sqrt{\gamma R_g T} \frac{v}{\sqrt{\gamma R_g T}} = \left(\frac{\gamma W}{T}\right)^{1/2} p M$$

where $R_g = R/W$ is the specific gas constant, R denotes the molar gas constant, W is the molecular weight of the gas mixture, T stands for the local (random

[1] With appropriate changes in wording, the present analysis applies also to a throatless motor.

translational) temperature, and M identifies the local MACH number. Hence the continuity equation becomes

$$\left(\frac{\gamma_1 W_1}{T_1}\right)^{1/2} p_1 M_1 = \left(\frac{\gamma_2 W_2}{T_2}\right)^{1/2} p_2 M_2. \tag{1}$$

Similarly, the equation for conservation of momentum is

$$p_1 + \varrho_1 v_1^2 = p_2 + \varrho_2 v_2^2$$

where

$$p + \varrho v^2 = p\left(1 + \frac{\varrho}{p} \frac{v^2}{\gamma R_g T} \gamma R_g T\right) = p(1 + \gamma M^2)$$

whence

$$p_1(1 + \gamma_1 M_1^2) = p_2(1 + \gamma_2 M_2^2). \tag{2}$$

In terms of an integral over stagnation temperatures and the total heat release per unit mass q, the energy equation reduces to

$$q = \int_{T_{s1}}^{T_{s2}} c_p \, dT_s \tag{3}$$

where c_p is the specific heat at constant pressure.

The stagnation temperature T_s and the translational temperature T are related through the expression

$$T_s = T + \frac{v^2}{2 c_p}$$

or

$$T_s = T\left(1 + \frac{v^2}{2 c_p T} \frac{\gamma R_g T}{\gamma R_g T}\right) = T\left(1 + \frac{\gamma R_g}{2 c_p} M^2\right) = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

since $R_g = c_p - c_v$ and $\gamma = c_p/c_v$. Hence Eq. (1) becomes

$$\left(\frac{\gamma_1 W_1}{T_{s1}}\right)^{1/2} p_1 M_1 \left(1 + \frac{\gamma_1 - 1}{2} M_1^2\right)^{1/2} = \left(\frac{\gamma_2 W_2}{T_{s2}}\right)^{1/2} p_2 M_2 \left(1 + \frac{\gamma_2 - 1}{2} M_2^2\right)^{1/2}.$$

We may now eliminate the pressure between the continuity and momentum equations by dividing this last relation by Eq. (2). We then find that

$$U_2(\gamma_2, M_2 = 1) = U_1(\gamma_1, M_1) \left(\frac{T_{s2}}{T_{s1}} \frac{W_1}{W_2}\right)^{1/2} \tag{4}$$

where

$$U = \frac{\gamma^{1/2} M \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/2}}{1 + \gamma M^2}. \tag{5}$$

The function U has been tabulated in [3] for various values of γ and M.

For specified values of the heat release and c_p, the change in stagnation temperature and, therefore, the stagnation temperature ratio, are defined by Eq. (3). Hence, Eq. (4) may be used to determine M_1 for known values of γ_1, γ_2, W_1 and W_2. The stagnation pressure loss may then be evaluated by using the expression

$$\frac{p_{s2}}{p_{s1}} = \left(\frac{p_{s2}}{p_2}\right)\left(\frac{p_1}{p_{s1}}\right)\left(\frac{p_2}{p_1}\right) = \left(\frac{p_{s2}}{p_2}\right)\left(\frac{p_1}{p_{s1}}\right)\left(\frac{1 + \gamma_1 M_1^2}{1 + \gamma_2}\right) \tag{6}$$

since $M_2 = 1$. The pressure ratios (p_{s2}/p_2) and (p_1/p_{s1}) are conveniently obtained by using existing tabulations for isentropic expansions [4]. For these calculations it is actually not necessary to utilize constant values of W and γ.

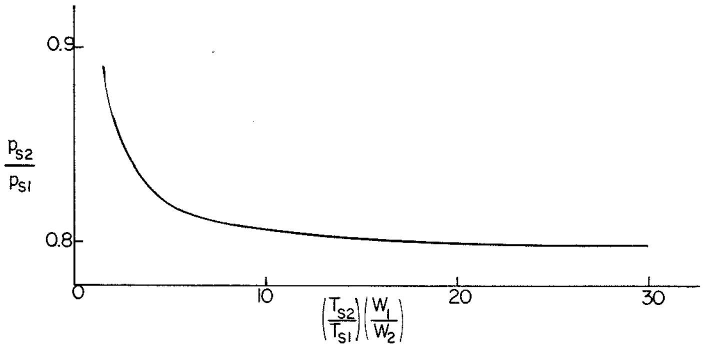

Fig. 2. The stagnation pressure ratio p_{s2}/p_{s1} as a function of the parameter (T_{s2}/T_{s1}) (W_1/W_2) for a diverging rocket ($\gamma_1 = \gamma_2 = 1.30$)

The stagnation pressure ratio p_{s2}/p_{s1} has been calculated as a function of the parameter (T_{s2}/T_{s1}) (W_1/W_2) from Eq. (6) for $\gamma_1 = \gamma_2 = 1.3$ by using Eq. (4) for the determination of M_1. The results are plotted in Fig. 2. Reference to Fig. 2 shows that $p_{s2}/p_{s1} \cong 0.8$ for (T_{s2}/T_{s1}) (W_1/W_2) greater than about 10; additional data may be obtained from [6].

2. Definitions of the Thrust Coefficient for Diverging and for Conventional Rocket Engines

The thrust coefficient $C_{F,\mathrm{con}}$ for a conventional rocket engine (see Fig. 3) is defined by the expression

$$C_{F,\mathrm{con}} = \frac{F}{p_{s0} A_*} \tag{7}$$

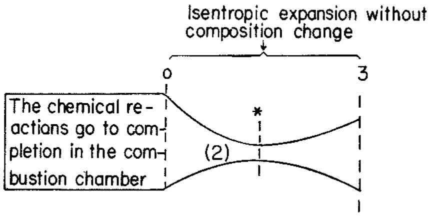

Fig. 3. Schematic diagram of a conventional rocket engine

where F is the thrust on the engine, p_{s0} identifies the chamber pressure, and A_* is the cross-sectional area at the nozzle throat. For properly expanded nozzles with p_3 equal to the external pressure,

$$F = (\varrho_3 v_3 A_3) v_3 = A_3 p_3 \gamma_2 M_3{}^2 \tag{8}$$

if γ_2 denotes the (constant) heat capacity ratio for the expanding gases. We note that in a conventional rocket

$$p_{s0} = p_{s3}$$

if we neglect dissipative terms during expansion and chemical reactions during nozzle flow. Thus

$$C_{F,\mathrm{con}} = \frac{\gamma_2 M_3{}^2 (A_3/A_*)}{(p_{s0}/p_3)}. \tag{9}$$

For the diverging reactor we define a thrust coefficient by the analogous expression

$$C_{F,\,\mathrm{div}} = \frac{F}{p_{s2}\,A_{1,2}} = \frac{\gamma_2\,M_3^{\,2}(A_3/A_{1,2})}{(p_{s3}/p_3)}\,. \tag{10}$$

But

$$\frac{p_{s3}}{p_3} = \frac{p_{s2}}{p_3} = \left(\frac{p_{s2}}{p_{s1}}\right)\left(\frac{p_{s1}}{p_3}\right) = \left(\frac{p_{s2}}{p_{s1}}\right)\left(\frac{p_{s0}}{p_3}\right)$$

whence

$$C_{F,\,\mathrm{div}} = \frac{\gamma_2\,M_3^{\,2}(A_3/A_{1,2})}{\left(\dfrac{p_{s2}}{p_{s1}}\right)\left(\dfrac{p_{s0}}{p_3}\right)}\,. \tag{11}$$

It is interesting to observe that Eqs. (9) and (11) differ formally only through the occurrence of the term (p_{s2}/p_{s1}) in the denominator of Eq. (11) since A_* and $A_{1,2}$ are analogous quantities. This pressure ratio, in turn, has been related to the heat release through $(T_{s2}/T_{s1})\,(W_1/W_2)$ in the preceding Section II 1.

3. Performance Evaluations for the Diverging Reactor

It is clearly possible to make a number of different performance evaluations for the diverging rocket. The performance estimate will then depend somewhat

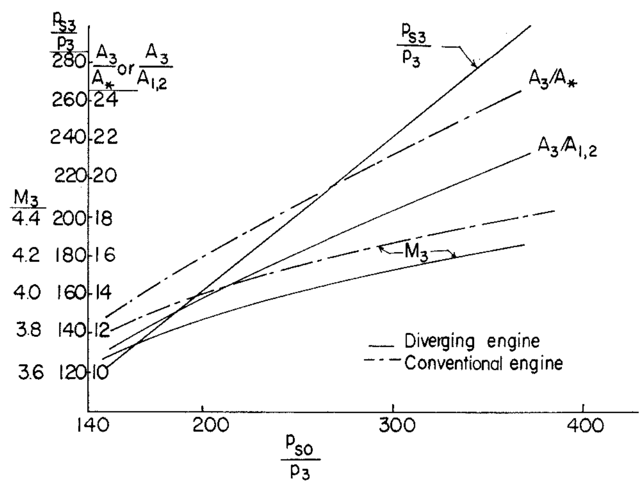

Fig. 4. The quantities M_3 and A_3/A_* (for a conventional rocket) and M_3, p_{s3}/p_3 and $A_3/A_{1,2}$ (for a diverging rocket) as a function of the pressure ratio p_{s0}/p_3 [$(T_{s2}/T_{s1})\,(W_1/W_2) = 10$ for the diverging rocket, $\gamma_2 = 1.30$]

on the predetermined design restrictions. We shall now consider two instructive examples. We restrict our discussion to the representative case with $\gamma_2 = 1.30$ and $(T_{s2}/T_{s1})\,(W_1/W_2) > 10$ for which Fig. 2 shows that $p_{s2}/p_{s1} \simeq 0.80$. The numerical work is facilitated by constructing a universal plot in which p_{s3}/p_3, A_3/A_*, $A_3/A_{1,2}$, and M_3 are related to the pressure ratio (cf. Fig. 4). For conventional rockets this calculation involves only the use of tabulated isentropic functions which first yield M_3 through the expression

$$\frac{p_{s0}}{p_3} = \left(1 + \frac{\gamma - 1}{2} M_3^2\right)^{\gamma/(\gamma - 1)}$$

and then A_3/A_* in terms of M_3 [4]. The curve $p_{s3}/p_3 = p_{s0}/p_3$ for the conventional engine is not plotted in Fig. 4. For the diverging rocket it is convenient to compute first the ratio p_{s3}/p_3 and then to evaluate M_3 and $A_3/A_{1,2}$ through use of the isentropic expansion relations.

III. Comparison of Engines with Fixed Thrust and Pressure Ratio p_{s0}/p_3

Consider a conventional and a diverging engine both of which deliver the thrust $F = 25{,}000$ lbs and which are designed with pressure ratios $p_{s0}/p_3 = 200$. Also assume that $p_3 = 1$ psia and $(T_{s2}/T_{s1})\,(W_1/W_2) > 10$.

For the conventional engine we find from Fig. 4 that $M_3 = 4.0$ and $A_3/A_* = 15.9$. Hence, using Eq. (9), $C_{F,\mathrm{con}} = 1.63$; it now follows from Eq. (7) that

$$A_* = \frac{2.5 \times 10^4}{1.64 \times 200} = 76.3 \text{ in}^2$$

and

$$A_3 = 15.9\, A_* = 1211 \text{ in}^2.$$

Similarly, for the diverging rocket, $M_3 = 3.87$, $A_3/A_{1,2} = 13.8$, $p_{s3}/p_3 = 162$, $C_{F,\mathrm{div}} = 1.64$ [from Eq. (10)],

$$A_{1,2} = \frac{2.5 \times 10^4}{1.63 \times 162} = 94.8 \text{ in}^2,$$

and

$$A_3 = 13.8\, A_{1,2} = 1308 \text{ in}^2.$$

Using customary design estimates, the weight ratio of the diverging nozzle sections for the two engines [5] is given by the relation

$$\frac{m^n_{\mathrm{div}}}{m^n_{\mathrm{con}}} = \frac{\left(\dfrac{A_3}{A_{1,2}}\dfrac{F}{p_{s3}}\right)_{\mathrm{div}}}{\left(\dfrac{A_3}{A_*}\dfrac{F}{p_{s3}}\right)_{\mathrm{conv}}} = 1.07.$$

Hence it follows that the diverging rocket will be relatively lighter provided the chamber and converging sections of the conventional engine have more than about 7% of the weight of the diverging section and the effective weight of the diverging reactor between the injector and reaction planes is negligibly small. We expect in practice that the combined weights of the motor and converging sections of a conventional engine will constitute about 10% of the total weight. Hence it follows that optimal design of the diverging reactor will be roughly comparable with the usually achieved design of a conventional rocket for equivalent engine thrust and pressure ratio p_{s0}/p_3.

IV. Comparison of Engines with Identical Diverging Sections but Different Pressure Ratios p_{s0}/p_3

Consider two engines which develop equivalent thrust but with

$$\frac{A_3}{A_*} = \frac{A_3}{A_{1,2}} = 15.9.$$

Then, from Fig. 4, $p_{s0} = 244$ psia for the diverging reactor whereas $p_{s0} = 200$ psia for the conventional engine. The corresponding weight ratio for the pumping equipment [5] is

$$\frac{m^p_{\text{div, pump}}}{m^p_{\text{conv, pump}}} \simeq \frac{(p_{s0})_{\text{div}}{}^{2/3}}{(p_{s0})_{\text{conv}}{}^{2/3}} = 1.14.$$

Therefore, if the chamber and converging section of the conventional engine weigh about 14% more than the pumping equipment, then the diverging rocket will be the lighter engine if we neglect the influence of the slightly higher injector end pressure on the weight of the diverging reactor and, furthermore, assume again that the chemical reactions are completed very close to the injector plane.

Comparison of the preceding two examples suggests that the use of relatively large pressure ratios p_{s0}/p_3 for the diverging rocket is impractical since the pumping equipment makes a major contribution to the mass of the engine.

V. Conclusions

Examination of the performance data specified in the preceding paragraph shows that diverging and conventional rocket engines should yield comparable facilities after optimum development of the relatively untried diverging rocket. It is difficult to assess the practical utility of the new device since it possesses both an obvious advantage and an obvious disadvantage. The advantage lies in the possibility of utilizing very hot, high-energy, propellant systems under conditions in which excessive temperatures and heat losses are not encountered because the chemical energy may be transformed directly into translational energy and because the diverging rocket may be relatively easier to cool[1]. On the other hand, in order to approximate our assumed combustion model (compare Fig. 1), it is apparent that the diverging reactor must give very efficient combustion near the injector plane, i.e., severe heat transfer and erosion should occur when the practical performance of the device approximates optimum design.

In conclusion it is appropriate to speculate on the possible existence of high-frequency instabilities in diverging reactors for which neither experimental nor theoretical data are available. The nature of the design actually appears to be rather favorable for the suppression of high-frequency instabilities: transverse modes should be damped out because of the rapidly changing cross-sectional area of a device in which strong composition inhomogeneities must accompany chemical changes; furthermore, it is probably impossible to sustain longitudinal oscillations because of extreme damping which must accompany diverging, supersonic, flow with chemical reactions. On the basis of the preceding considerations, it appears highly desirable to initiate a large-scale engine program for a high-energy liquid propellant mixture (e.g., ClF_3-N_2H_4). It is unlikely that small-scale engine tests will provide significant information either concerning the practical difficulties inherent in proper injector design or concerning the nature of combustion instabilities.

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[1] A practical diverging rocket design might well involve the use of ablating material near the injector end. The performance of the engine should not suffer significantly because of enlargement of cross-sectional area near the injector associated with ablation.

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